



# New binary and ternary quasi-cyclic codes with good properties

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## Abstract

One of the most important and challenging problems in coding theory is to construct codes with best possible parameters and properties. The class of quasi-cyclic (QC) codes is known to be fertile to produce such codes. Focusing on QC codes over the binary field, we have found 113 binary QC codes that are new among the class of QC codes using an implementation of a fast cyclic partitioning algorithm and the highly effective ASR algorithm. Moreover, these codes have the following additional properties: a) they have the same parameters as best known linear codes, and b) many of the have additional desired properties such as being reversible, LCD, self-orthogonal or dual-containing. Additionally, we present an algorithm for the generation of new codes from QC codes using ConstructionX, and introduce 33 new record breaking linear codes over  $GF(2)$ ,  $GF(3)$  and  $GF(5)$  produced from this method.

**Keywords** Quasi-cyclic codes · Best known codes · Reversible codes · LCD codes · Self-orthogonal codes

**Mathematics Subject Classification** 94B05 · 94B60 · 94B65

## 1 Introduction and motivation

A linear block code  $C$  of length  $n$  over the finite field  $GF(q)$  (the code alphabet, also denoted by  $\mathbb{F}_q$ ) is a vector subspace of  $\mathbb{F}_q^n$ . If the dimension of  $C$  is  $k$  and its minimum distance is  $d$ , then  $C$  is referred to as an  $[n, k, d]_q$ -code. Elements of  $C$  are called codewords. A matrix whose rows constitute a basis for  $C$  is called a generator matrix of  $C$ .

One of the main goals of coding theory is to construct codes with best possible parameters. This is an optimization problem that can be formulated in a few different ways. For example, we can fix  $n$  and  $k$  (hence the information rate of the code) and ask for the largest possible value  $d_q[n, k]$  of  $d$ . A code of length  $n$  and dimension  $k$  whose minimum distance is  $d_q[n, k]$

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is an optimal code (or a distance-optimal code). Similarly, one can fix any two of the three parameters of a linear code and look for the optimal value of the third parameter. There are many theoretical bounds on the parameters of a linear code. There are also databases of best known linear codes (BKLC). The online database (Grassl 2022) is well known in the coding theory research community. For a given set of values for  $q$ ,  $n$ , and  $k$ , the database gives information about

- a) the best theoretical upper bound for  $d$ ,
- b) the highest minimum distance of a best known linear code with the given length and the dimension which is a lower bound on  $d$ .

Lower bounds are usually obtained through explicit constructions. A code with an explicit construction that has the largest known minimum distance for these parameters provides a lower bound. It is possible to obtain codes with the same parameters in multiple ways and with different structures. In most cases, there are gaps between theoretical upper bounds and lower bounds. The Magma software (2022) also has a similar database. Additionally, there are also more specialized databases such as the one specifically for quasi-cyclic (QC) and quasi-twisted (QT) codes (Chen 2022).

Constructing codes with best possible parameters is a challenging problem. It is clear from the databases that in most cases optimal codes are not yet known. They are usually known when either  $k$  or  $n - k$  is small. Hence, there are many instances of this optimization problem that are open. For example, for  $q = 5$  and  $n = 100$ , there are gaps between the upper bounds and lower bounds on  $d$  for every dimension  $5 \leq k \leq 91$ .

There are two main reasons why this optimization problem is very challenging even with the help of modern computers. First, determining the minimum distance of a linear code is computationally intractable (Vardy 1997), so it takes significant amount of time to find the minimum distance of a single code when the dimension is large and becomes infeasible after a certain point. Second, for a given length and dimension, the number  $\frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}$  of linear codes of length  $n$  and dimension  $k$  over  $\mathbb{F}_q$  is large and grows quickly. Hence, an exhaustive computer search on linear codes is not feasible. Therefore, researchers focus on specific classes of codes with rich mathematical structures that are known to contain many codes with good parameters. The class of quasi-cyclic (QC) codes has an excellent record of producing many codes with best known parameters. They are the focus of our search in this work. In fact, we have combined three methods that are known to be useful in constructing codes with good parameters.

- a) The ASR search algorithm for QC codes
- b) Using a recently introduced algorithm to test equivalence of cyclic codes
- c) ConstructionX

Our search revealed 113 binary QC codes that are new among the class of QC codes according to Grassl (2022). Moreover, our codes also have the following additional features:

- Each of these codes has the same parameters as BKLCs in Grassl (2022).
- In many cases, the BKLCs in the database (Grassl 2022) have indirect, multi-step constructions so it is more efficient and desirable to obtain them in the form of QC codes instead.
- Many of our codes have additional desirable properties, such as being self-orthogonal, reversible, or linear complementary dual (LCD).

Additionally, we applied the ConstructionX method that uses QC codes with good parameters as the input and found 33 new linear codes of which 8 are binary.

The rest of the paper is organized as follows. In the next section, we present some basic definitions that are fundamental to our work. In section 3, we explain our search method. Finally, we list the new codes in the last section.

## 2 Basic definitions

Cyclic codes have a prominent place in coding theory for both theoretical and practical reasons. Some of the best known examples of codes are either cyclic or equivalent to cyclic codes including binary Hamming codes, the Golay codes, BCH codes, Reed–Solomon codes, and quadratic residue codes to name a few. They are conveniently implemented via shift registers. Theoretically, they establish a key link between coding theory and algebra. The first step in this connection is to represent a vector  $(c_0, c_1, \dots, c_{n-1})$  in  $\mathbb{F}_q^n$  as the polynomial  $c(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1}$  of degree less than  $n$ . This correspondence defines a vector space isomorphism between  $\mathbb{F}_q^n$  and the set of polynomials of degree  $< n$  over  $\mathbb{F}_q$ . With this identification, we use vectors/codewords and polynomials interchangeably.

**Definition 2.1** A linear code  $C$  is called cyclic if it is closed under the cyclic shift  $\pi$  on  $\mathbb{F}_q^n$ , i.e. whenever  $c = (c_0, c_1, \dots, c_{n-1})$  is a codeword of  $C$ , then so is  $\pi(c) = (c_{n-1}, c_0, \dots, c_{n-2})$ .

In the polynomial representation, the cyclic shift of a codeword  $c(x)$  corresponds to  $xc(x) \pmod{x^n - 1}$ . It follows that a cyclic code is an ideal in the quotient ring  $\mathbb{F}_q[x]/\langle x^n - 1 \rangle$  which is a principal ideal ring. Hence, any cyclic code  $C$  can be viewed as a principal ideal  $C = \langle g(x) \rangle = \{f(x)g(x) \pmod{x^n - 1} : f(x) \in \mathbb{F}_q[x]\}$  generated by  $g(x)$ . A cyclic code  $C$  has many generator polynomials and among them is a unique one. The monic, non-zero polynomial of least degree in  $C$  is a unique generator for  $C$ . We will refer to this unique generator as the (standard) generator of  $C$ . When speak of “the generator polynomial” of a cyclic code, the standard generator should be understood.

The following are well known about cyclic codes.

**Lemma 2.1** Let  $C = \langle g(x) \rangle$  be a cyclic code of length  $n$  over  $\mathbb{F}_q$  where  $g(x)$  is the standard generator polynomial. Then the following holds

- (1)  $g(x)$  is a divisor of  $x^n - 1$  over  $\mathbb{F}_q$ . Hence  $x^n - 1 = g(x)h(x)$  for some  $h(x) \in \mathbb{F}_q[x]$ .
- (2) The polynomial  $h(x)$  is called the check polynomial and it has the property that a word  $v(x)$  is in  $C$  if and only if  $h(x)v(x) = 0$  in  $\mathbb{F}_q[x]/\langle x^n - 1 \rangle$ .
- (3) The dimension of  $C$  is  $k = n - \deg(g(x)) = \deg(h(x))$  and a basis for  $C$  is  $\{g(x), xg(x), \dots, x^{k-1}g(x)\}$ .
- (4) If  $g(x) = g_0 + g_1x + \dots + g_r x^r$  then  $g_0 \neq 0$  and the following circulant matrix is a generator matrix for  $C$ , where each row is a cyclic shift of the previous row.

$$G = \begin{bmatrix} g_0 & g_1 & g_2 & \dots & g_r & 0 & 0 & \dots & 0 \\ 0 & g_0 & g_1 & g_2 & \dots & g_r & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & g_0 & g_1 & \dots & g_r \end{bmatrix}$$

- (5)  $C = \langle p(x) \rangle$  if and only if  $p(x) = f(x)g(x)$  where  $\gcd(f(x), h(x)) = 1$ .
- (6) There is a one-to-one correspondence between divisors of  $x^n - 1$  and cyclic codes of length  $n$  over  $\mathbb{F}_q$ .

In the study of cyclic codes, the notion of cyclotomic cosets is fundamental. There is a one-to-one correspondence between cyclotomic cosets of  $q \bmod n$  and irreducible divisors of  $x^n - 1$  over  $\mathbb{F}_q$ . The important class of BCH codes is defined based on cyclotomic cosets. We will make use of cyclotomic cosets in our search process.

Cyclic codes have many useful generalizations. One of the most important generalizations is quasi-cyclic (QC) codes where we can shift coordinates of codewords by more than one positions.

**Definition 2.2** A linear code  $C$  is said to be  $\ell$ -quasi-cyclic (QC) if for a positive integer  $\ell$ , whenever  $c = (c_0, c_1, \dots, c_{n-1})$  is a codeword, so is  $(c_{n-\ell}, c_{n-\ell+1}, \dots, c_{n-1}, c_0, c_1, \dots, c_{n-\ell-2}, c_{n-\ell-1})$ . Such a code is called a QC code of index  $\ell$ , or an  $\ell$ -QC code.

It is well known that the length of a QC code must be a multiple of  $\ell$ , hence  $n = m\ell$  for some positive integer  $m$  (Aydin et al. 2001). Let  $R_m = \frac{\mathbb{F}_q[x]}{\langle x^m - 1 \rangle}$ . Then an  $\ell$ -QC code is an  $\mathbb{F}_q[x]$ -sub-module of  $R_m^\ell$  (Aydin et al. 2001). QC codes are known to contain many codes with good parameters. Hundreds of BKLCs in Grassl (2022) are obtained from QC codes. A particularly effective search algorithm called ASR was presented in Aydin et al. (2001) and has been employed many times since then (e.g., Daskalov and Hristov 2003; Ackerman and Aydin 2011; Aydin et al. 2017, 2020). This is the basis of the method we use in this work as well.

For any linear code  $C$ , its dual code is defined as  $C^\perp = \{v \in \mathbb{F}_q^n : v \cdot c = 0 \text{ for all } c \in C\}$  where  $v \cdot c$  is the standard inner product in  $\mathbb{F}_q^n$ . If the dimension of  $C$  is  $k$ , then the dimension of  $C^\perp$  is  $n - k$ . A code  $C$  is self-orthogonal if  $C \subseteq C^\perp$ , i.e., for any two codewords  $a, b \in C$ ,  $a \cdot b = 0$ . An  $[n, k]_q$  code  $C$  is self-dual if  $C = C^\perp$ . Note that in this case, the dimensions of  $C$  and  $C^\perp$  need to be equal. Thus,  $k = n/2$ . A code  $C$  is dual-containing if  $C^\perp \subseteq C$ .

Self-dual codes are an important area of research and there is a vast literature about them in coding theory. One application of self-orthogonal codes is in constructing quantum error correcting codes (QECC) from classical codes. A method of constructing quantum error correcting codes (QECC) from classical codes was given in Calderbank et al. (1998). Since then researchers have investigated various methods of using classical error correcting codes to construct new QECCs. The majority of the methods have been based on the CSS construction given in Calderbank et al. (1998). In this method, self-dual, self-orthogonal and dual-containing linear codes are used to construct quantum codes. The CSS construction requires two linear codes  $C_1$  and  $C_2$  such that  $C_2^\perp \subseteq C_1$ . Hence, if  $C_1$  is a self-dual code, then we can construct a CSS quantum code using  $C_1$  alone since  $C_1^\perp \subseteq C_1$ . If  $C_1$  is self-orthogonal, then we can construct a CSS quantum code with  $C_1^\perp$  and  $C_1$  since  $C_1 \subseteq C_1^\perp$ . Similarly in the case  $C_1$  is a dual-containing code. In a recent work, many new best known quantum codes (BKQC) have been found from classical self-orthogonal codes (Aydin et al. 2020).

QC codes generated by the ASR algorithm (Aydin et al. 2001) use good cyclic codes as building blocks. For the codes generated by this algorithm, the length is typically much larger than the dimension. Thus, in most cases, we have  $k < n/2$ . Consequently, we can only find self-orthogonal codes from this method which can still be used in constructing quantum codes.

A code  $C$  is linear complementary dual (LCD) if  $C \cap C^\perp = \{0\}$ . They were first introduced by Massey (Massey 1992), and were seen to have an optimal solution for a two-user binary adder channel as well as decoding algorithms that are less complex than those for general linear codes. They are also useful in cryptography by protecting the information managed by

sensitive devices, particularly against fault invasive attacks and side-channel attacks (SCA) (Lu et al. 2020). We have been able to find a number of QC codes that are LCD in our search.

Another useful property of a code is being reversible. A code  $C$  is reversible if for any codeword  $(w_0, w_1, \dots, w_{n-2}, w_{n-1}) \in C$  its reverse  $(w_{n-1}, w_{n-2}, \dots, w_1, w_0)$  is also in  $C$ . Suppose we have a reversible code  $C$  stored in some storage medium. Since the reverse of every codeword of  $C$  is also a codeword, the stored data can be read from either end of the code, which could be advantageous if, for example, we are interested in only the information at one end of the code. If the decoder has to read the entire code before beginning the decoding process, then the code being reversible is not important. However, if the code can be decoded digit by digit, then the same decoding circuit can be used irrespective of the end of the code that is fed first (Massey 1964).

### 3 The ASR search method for QC codes

Our goal in this search was to find binary QC codes with good parameters and good properties. We employed the generalized version of the ASR algorithm described in Aydin et al. (2019) as our search method. The first step in the search process is to obtain all cyclic codes for all lengths of interest and partition them into equivalence classes based on code equivalence. Although Magma software has a command to test equivalence of linear codes, a more efficient method that is specifically for cyclic codes has recently been introduced in Aydin and VandenBerg (2021). We implemented this algorithm in our work. The rest of this section gives more details on the search process.

#### 3.1 Cyclic partition algorithm

The generalized ASR algorithm is based on the notion of equivalent codes.

**Definition 3.1** Two linear codes are equivalent if one can be obtained from the other by any combinations of the following transformations:

- (1) A permutation of coordinates.
- (2) Multiplication of elements in a fixed position by a non-zero scalar in  $\mathbb{F}_q$ .
- (3) Applying an automorphism of  $\mathbb{F}_q$  to each component of the vectors.

If only the first transformation is used, then the resulting codes are called permutation equivalent. This is a very important special case and in fact, for binary codes, it is the only type of code equivalence that is possible.

The algorithm given in Aydin and VandenBerg (2021) allows us to partition cyclic codes of a given length into equivalence classes, and then choose one code from each class, significantly reducing the computational workload and allowing us to quickly move on to QC construction.

The algorithm to generate all binary cyclic codes of a given length that are not equivalent to each other is as follows:

- (1) Start with a length  $n$ .
- (2) Write  $n$  in the form  $n'2^t$  such that  $n'$  is not divisible by 2.
- (3) Generate cyclotomic cosets of 2 mod  $n'$ .
- (4) Generate all multisets of the cosets such that each coset can be repeated up to  $2^t$  times.

- (5) Check for linear maps between the generated multisets of the same size. If a linear map exists between two multisets, then we know that the codes defined by them are equivalent so we eliminate one of them.
- (6) Find  $\beta$ , a primitive  $n'$ -th root of unity over  $\mathbb{F}_2$ .
- (7) Use the remaining multisets and  $\beta$  to obtain generator polynomials such that a multiset  $w = \{w_0, w_1, \dots, w_i\}$  generates the polynomial  $g_w(x) = (x - \beta^{w_0})(x - \beta^{w_1}) \dots (x - \beta^{w_i})$ .

As explained in Aydin and Vandenberg (2021), while in general it is possible for this algorithm to fail to distinguish between some equivalent codes, over the binary field, it is guaranteed to completely partition the code space. Checking the equivalence of two given linear codes is known to be equivalent to the graph isomorphism problem, which is believed to be NP Intermediate (Petrank and Roth 1997). Magma software has a function for this task for general linear codes, but it does not always work, and in many cases, it takes too long to finish. Our algorithm that is specifically for cyclic codes is much more efficient. We refer the reader to Aydin and Vandenberg (2021) for more on the details and performance of this algorithm. Having this algorithm to quickly produce cyclic codes, we could then move on to using the ASR search algorithm to generate QC codes. See Table 1.

### 3.2 The ASR algorithm

Given a generator polynomial  $p(x) = p_0 + p_1x + \dots + p_{m-1}x^{m-1}$  (not necessarily the standard generator) of a cyclic code  $C$  of length  $m$  over finite field  $\mathbb{F}_q$ ,  $C$  has a generator matrix of the following form:

$$G = \begin{bmatrix} p_0 & p_1 & p_2 & \dots & p_{m-1} \\ p_{m-1} & p_0 & p_1 & \dots & p_{m-2} \\ p_{m-2} & p_{m-1} & p_0 & \dots & p_{m-3} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{m-k+1} & p_{m-k+2} & p_{m-k+2} & \dots & p_{m-k} \end{bmatrix}$$

Such a matrix is called a circulant matrix (Aydin et al. 2020). As a generalization of cyclic codes, a generator matrix of a QC code consists of blocks of circulant matrices. In general, a generator matrix of an  $\ell$ -QC code has the following form

$$G = \begin{bmatrix} G_{11} & G_{12} & \dots & G_{1\ell} \\ G_{21} & G_{22} & \dots & G_{2\ell} \\ \vdots & \vdots & & \vdots \\ G_{r1} & G_{r2} & \dots & G_{r\ell} \end{bmatrix}$$

where each  $G_{ij}$  is a circulant matrix corresponding to a cyclic code. Such a code is called an  $r$ -generator QC code. (Aydin et al. 2020) The case  $[G_1 \ G_2 \ \dots \ G_\ell]$  gives 1-generator QC codes, which is the case we have considered in this work.

We begin the ASR search algorithm by taking one generator  $g(x)$  of a cyclic code of length  $m$  from each equivalence class. Then, we construct the generator of an  $\ell$ -QC code in the form

$$(f_1(x)g(x), f_2(x)g(x), \dots, f_\ell(x)g(x)),$$

**Table 1** New linear complementary dual codes

[195, 38, 58] <sub>2</sub>	3 [6135332413622, 5511163061413, 0760162305021, 41167113347, 1304422614561]
[172, 42, 46] <sub>2</sub>	3 [16305255301422, 12552443046071, 26367761131227, 56712075265061]
[170, 32, 52] <sub>2</sub>	5 [15017544142, 62327171172, 5656706772, 54766743242, 50452552233]
[165, 32, 50] <sub>2</sub>	3 [7451352027, 7530304175, 16125740433, 70670753002, 2255573462]
[164, 40, 44] <sub>2</sub>	3 [0711720165373, 4162676126303, 731245452506, 4301603434632]
[160, 32, 48] <sub>2</sub>	104 [00542672771, 35714512512, 6146475721, 05016172172]
[156, 36, 44] <sub>2</sub>	11 [377230104735, 441101605347, 135525223756, 034664137676]
[153, 43, 38] <sub>2</sub>	727 [03362056173102, 273542165074071, 507007322260431]
[141, 46, 32] <sub>2</sub>	3 [340604477037257, 5555546265363011, 1447675154362301]
[138, 44, 32] <sub>2</sub>	5 [047572717636601, 042465541137103, 34632557665736]
[124, 30, 36] <sub>2</sub>	3 [3023466614, 104135571, 6774200266, 4032715624]
[122, 60, 20] <sub>2</sub>	3 [5714214006357625062, 32647415367136476374]
[120, 32, 32] <sub>2</sub>	104 [3211420427, 0605045201, 45670476662]
[117, 38, 28] <sub>2</sub>	3 [0466644754451, 2436541252673, 5131112141731]
[116, 28, 34] <sub>2</sub>	3 [442732531, 253642371, 2645552751, 4024662711]
[114, 54, 20] <sub>2</sub>	11 [350273450337664702, 105500762001021221]
[111, 36, 26] <sub>2</sub>	3 [515231504461, 200230765236, 126716101403]
[110, 50, 20] <sub>2</sub>	14 [03233320732245542, 03670445051210042]
[110, 40, 24] <sub>2</sub>	730471 [03646446602231, 6062203646014]
[108, 48, 20] <sub>2</sub>	101 [3300216514056443, 7000312523564625]
[108, 32, 28] <sub>2</sub>	12 [74160521111, 45604632562, 40250112373]
[105, 34, 26] <sub>2</sub>	3 [742252523401, 314437023031, 070422111261]
[105, 30, 28] <sub>2</sub>	14 [5153333215, 1365535577, 6626560476]
[105, 29, 28] <sub>2</sub>	771 [6162561672, 6651775572, 1064436731]
[105, 24, 32] <sub>2</sub>	7367 [51556275, 51764767, 20643705]
[104, 24, 32] <sub>2</sub>	5 [43441155, 53243002, 51421571, 06170414]
[102, 32, 26] <sub>2</sub>	5 [23016441171, 5155572505, 3261443103]
[100, 40, 20] <sub>2</sub>	1002 [3367605450137, 2264022063455]
[99, 32, 24] <sub>2</sub>	3 [65725410163, 17752117321, 5251344657]
[99, 31, 25] <sub>2</sub>	7 [23543624601, 1403477176, 00560656631]
[99, 30, 26] <sub>2</sub>	11 [0677144753, 1370221122, 461726547]
[99, 21, 32] <sub>2</sub>	57731 [2433502, 5112553, 6660032]
[94, 46, 16] <sub>2</sub>	3 [17570216336424, 6073617230441121]
[93, 30, 24] <sub>2</sub>	3 [7475563176, 4415177161, 234165126]
[90, 26, 24] <sub>2</sub>	52 [531435171, 114350132, 353040702]
[90, 24, 26] <sub>2</sub>	101 [60122111, 60321712, 23436313]
[88, 20, 28] <sub>2</sub>	5 [3324751, 701554, 5053663, 4071362]
[84, 24, 24] <sub>2</sub>	12 [37211625, 33260411, 43757466]
[84, 18, 28] <sub>2</sub>	11 [234021, 320241, 060265, 172327]
[82, 20, 26] <sub>2</sub>	70215061 [7157701, 570521]
[81, 19, 26] <sub>2</sub>	777 [726367, 21155, 303002]
[78, 24, 22] <sub>2</sub>	5 [27512541, 02121473, 60544261]

**Table 1** continued

$[70, 28, 16]_2$	102	[324234467, 0654036141]
$[63, 20, 18]_2$	3	[4312562, 4302502, 5233323]
$[63, 13, 24]_2$	575	[04021, 5456, 67011]
$[62, 30, 12]_2$	3	[3024406253, 4560604406]
$[60, 16, 20]_2$	12	[724711, 50616, 334001]
$[58, 28, 12]_2$	3	[4127557501, 402073244]
$[56, 24, 12]_2$	12	[67163116, 53054674]
$[52, 24, 12]_2$	5	[7360021, 5267555]
$[51, 16, 16]_2$	3	[623251, 7054, 15702]
$[50, 20, 12]_2$	14	[0150542, 3034521]
$[34, 16, 8]_2$	3	[71404, 010461]

where all  $f_i(x)$ 's are chosen arbitrarily from  $\mathbb{F}_q[x]/(x^m - 1)$  such that they are relatively prime to  $h(x)$ , the check polynomial of the cyclic code generated by  $g(x)$ , and  $\deg(f_i(x)) < \deg(h(x))$ . The following theorem is the basis of the ASR algorithm.

**Theorem 3.1** (Aydin et al. 2001) *Let  $C$  be a 1-generator  $\ell$ -QC code over  $\mathbb{F}_q$  of length  $n = m\ell$  with a generator  $G(x)$  of the form:*

$$G(x) = (f_1(x)g(x), f_2(x)g(x), \dots, f_\ell(x)g(x)),$$

where  $x^m - 1 = g(x)h(x)$  and for all  $i = 1, \dots, \ell$ ,  $\gcd(h(x), f_i(x)) = 1$ . Then,  $C$  is an  $[n, k, d']_q$ -code where  $k = m - \deg(g(x))$ , and  $d' \geq \ell \cdot d$ ,  $d$  being the minimum distance of the cyclic code  $C_g$  of length  $m$  generated by  $g(x)$ .

## 4 New binary QC codes

Using this search method, we have been able to generate 113 binary QC codes with the following features.

- (1) Every one of these codes is new among the class of binary QC codes according to the database (Chen 2022).
- (2) Each of these codes has the same parameters as BKLCs in Grassl (2022).
- (3) In many cases, the BKLCs in the database (Grassl 2022) have indirect, multi-step constructions so it is more efficient and desirable to obtain them in the form of QC codes instead. All of the codes we present are QC and in many cases the corresponding codes in Grassl (2022) do not have simple constructions.
- (4) A number of our codes have additional desirable properties, such as being self-orthogonal, reversible, and linear complementary dual (LCD).

The following table lists the parameters, properties, and generators of these new QC codes. The generators are listed by their coefficients in base 8 for a compact representation. For example, consider the length  $n = 70$  code in Table 2 below whose generator  $g$  is  $1 + x^2 + x^3 + x^4$ . The coefficients of this polynomial are 10111 in increasing powers of  $x$  from left to right. We break this up into blocks of three, so it becomes 101,110. These blocks are then converted to base 8, reading left to right, so they become 53. The number



**Table 2** New self-orthogonal codes

[213, 35, 68] <sub>2</sub>	7120103605521	[111124074763, 432070306671, 774512423733]
[158, 39, 44] <sub>2</sub>	35441216370232	[5172674573162, 1174545113363]
[155, 10, 72] <sub>2</sub>	16502701	[554, 4101, 57, 011, 7701]
[140, 16, 56] <sub>2</sub>	5033712	[605031, 630121, 253451, 376371]
[126, 11, 56] <sub>2</sub>	5232	[3761, 0012, 5523, 5302, 1202, 454]
[120, 18, 44] <sub>2</sub>	52101	[241454, 375415, 221446, 331264]
[112, 19, 40] <sub>2</sub>	5021	[373773, 5774151, 356755, 1433301]
[112, 12, 48] <sub>2</sub>	124002	[4705, 7007, 5277, 0453]
[110, 34, 28] <sub>2</sub>	76653211	[60526046436, 202251730361]
[110, 24, 36] <sub>2</sub>	55500752743	[57222225, 34412755]
[110, 20, 4] <sub>2</sub>	376515033775	[3515263, 3706541]
[108, 21, 36] <sub>2</sub>	111	[5620102, 4110132, 2126317, 1604673]
[100, 21, 32] <sub>2</sub>	73	[0321052, 4140743, 1474414, 5140361]
[96, 20, 32] <sub>2</sub>	12	[140153, 6631273, 5771431, 4224433]
[96, 19, 32] <sub>2</sub>	55	[7522251, 653453, 631473, 627777]
[93, 15, 36] <sub>2</sub>	525412	[73036, 34767, 46131]
[92, 11, 40] <sub>2</sub>	73221	[3072, 046, 6641, 4673]
[92, 11, 40] <sub>2</sub>	73221	[3072, 046, 6641, 4673]
[90, 9, 40] <sub>2</sub>	3	[111, 551, 534, 455, 015, 615, 375, 645, 07]
[88, 21, 28] <sub>2</sub>	3	[0407703, 2145355, 7260573, 021165]
[84, 19, 28] <sub>2</sub>	7	[766644, 751675, 3504451, 551303]
[84, 15, 32] <sub>2</sub>	51153	[05752, 63554, 64613]
[84, 14, 32] <sub>2</sub>	102	[12042, 57752, 42473, 06653]
[80, 17, 28] <sub>2</sub>	71	[401643, 651151, 232501, 614002]
[80, 14, 32] <sub>2</sub>	341	[25232, 7, 0424, 6411]
[80, 13, 32] <sub>2</sub>	542	[41121, 4374, 65241, 3205]
[80, 12, 32] <sub>2</sub>	525	[5233, 5531, 4743, 534]
[78, 26, 20] <sub>2</sub>	70112	[000061552, 033006061]
[78, 12, 32] <sub>2</sub>	3625345231	[4532, 2146]
[75, 20, 24] <sub>2</sub>	14	[0027423, 726242, 3710732]
[70, 31, 16] <sub>2</sub>	53	[0215201037, 17453360511]
[70, 31, 16] <sub>2</sub>	53	[0215201037, 17453360511]
[69, 22, 20] <sub>2</sub>	3	[12325661, 6003045, 10405]
[66, 20, 20] <sub>2</sub>	5	[3343631, 027677, 0516553]
[64, 13, 24] <sub>2</sub>	71	[50421, 2344, 4624, 17661]
[62, 25, 16] <sub>2</sub>	131	[516317651, 45156344]
[60, 17, 20] <sub>2</sub>	71	[222761, 753102, 52627]
[54, 12, 20] <sub>2</sub>	101	[5233, 1144, 1726]
[46, 12, 16] <sub>2</sub>	5616	[5733, 6557]

**Table 3** New binary QC codes that are self-orthogonal and reversible

$[n, k, d]_q$	$g$	$[f_1, \dots, f_\ell]$
$[70, 30, 16]_2$	14	[1367566016, 5714137543]
$[52, 25, 12]_2$	3	[156307741, 752562251]

of  $f$  polynomials corresponds to the number of blocks, or index ( $\ell$ ) of the QC code, and  $m$  times  $\ell$  gives  $n$ , the total length. For this particular example,  $\ell = 2$ , hence the block length is  $m = 35$ . This means  $g(x)$  is a divisor of  $x^{35} - 1$ .

In addition to the additional properties that they possess, these new QC codes are usually better than the BKLCs currently listed in the database (Grassl 2022) for the reason that their constructions are far simpler. A QC code is more desirable than an arbitrary linear code for many reasons. It has a well understood algebraic structure and its generator matrix is determined by its first row alone. This property is being exploited in some cryptosystems that are based in coding theory to reduce the key sizes in McEliece type cryptosystems (Heyse et al. 2013). In comparison, the BKLCs in Grassl (2022) with the same parameters can have far more steps to construct. For example, we found a  $[213, 35, 68]_2$  code that has the same parameters as the comparable BKLC, as well as being self orthogonal and reversible. Being a QC code, it has a single step construction. The current record holder for the same length, dimension and field in Grassl (2022) on the other hand has a 17-step construction to achieve the same parameters and lacks any additional properties. In many cases, the codes presented here are simpler to construct with additional desirable properties and still having the same parameters as BKLCs. (See Table 3).

The codes that have additional properties are already listed in the tables above. Any new codes that are not listed in the tables above have their parameters recorded below. For the sake of space, we do not write down their generators. They are available from the authors. Moreover, these codes have been added to the database (Chen 2022) and their generators are available there as well.

### 5 A ConstructionX method for new codes from QC codes

In this section, we examine the construction of new good codes from existing QC codes using ConstructionX. This method was inspired by a similar work in Daskalov and Hristov (2017); however, we have generalized it and expanded on the details of the methodology.

ConstructionX is a method of creating new codes from existing good codes (Sloane et al. 1972, MacWilliams and Sloane 1977). Given a code  $C_1$  with parameters  $[n_1, k_1, d_1]$ , a sub-code  $C_2$  of it with parameters  $[n_1, k_1 - b, d_2]$ , and a third code  $C_3$  with parameters  $[n_2, b, d_3]$ , ConstructionX divides  $C_2$  into a union of cosets of  $C_1$  and attaches a different codeword of  $C_3$  to each coset. This results in a new code,  $C$ , which has parameters  $[n, k, d]$  such that  $n = n_1 + n_2, k = k_1$ , and  $d_2 \geq d \geq \min\{d_2, d_1 + d_3\}$ . Furthermore,  $C$  has a generator matrix of the form

$$\begin{bmatrix} G_1^* & G_3 \\ G_2 & 0 \end{bmatrix},$$

[218, 37, 68] <sub>2</sub>	[128, 31, 36] <sub>2</sub>	[108, 47, 20] <sub>2</sub>	[90, 34, 20] <sub>2</sub>
[200, 37, 60] <sub>2</sub>	[124, 51, 24] <sub>2</sub>	[108, 46, 20] <sub>2</sub>	[90, 28, 24] <sub>2</sub>
[190, 37, 56] <sub>2</sub>	[124, 50, 24] <sub>2</sub>	[108, 35, 26] <sub>2</sub>	[90, 27, 24] <sub>2</sub>
[175, 34, 52] <sub>2</sub>	[124, 37, 32] <sub>2</sub>	[108, 31, 28] <sub>2</sub>	[88, 32, 20] <sub>2</sub>
[168, 41, 44] <sub>2</sub>	[120, 58, 20] <sub>2</sub>	[108, 26, 32] <sub>2</sub>	[84, 31, 20] <sub>2</sub>
[160, 38, 44] <sub>2</sub>	[120, 49, 24] <sub>2</sub>	[105, 31, 28] <sub>2</sub>	[84, 30, 20] <sub>2</sub>
[160, 37, 44] <sub>2</sub>	[120, 47, 24] <sub>2</sub>	[102, 42, 20] <sub>2</sub>	[84, 23, 24] <sub>2</sub>
[160, 30, 50] <sub>2</sub>	[120, 46, 24] <sub>2</sub>	[102, 32, 26] <sub>2</sub>	[81, 26, 22] <sub>2</sub>
[153, 42, 38] <sub>2</sub>	[120, 39, 28] <sub>2</sub>	[100, 42, 20] <sub>2</sub>	[80, 37, 16] <sub>2</sub>
[150, 45, 36] <sub>2</sub>	[120, 33, 32] <sub>2</sub>	[100, 41, 20] <sub>2</sub>	[80, 28, 20] <sub>2</sub>
[150, 44, 36] <sub>2</sub>	[112, 52, 20] <sub>2</sub>	[96, 39, 20] <sub>2</sub>	[72, 31, 16] <sub>2</sub>
[144, 45, 34] <sub>2</sub>	[112, 51, 20] <sub>2</sub>	[96, 38, 20] <sub>2</sub>	[70, 29, 16] <sub>2</sub>
[138, 45, 32] <sub>2</sub>	[112, 50, 20] <sub>2</sub>	[96, 37, 20] <sub>2</sub>	[64, 29, 14] <sub>2</sub>
[132, 41, 32] <sub>2</sub>	[112, 47, 22] <sub>2</sub>	[96, 36, 20] <sub>2</sub>	[64, 25, 16] <sub>2</sub>
[132, 34, 36] <sub>2</sub>	[112, 27, 32] <sub>2</sub>	[96, 31, 24] <sub>2</sub>	[64, 24, 16] <sub>2</sub>
[132, 31, 38] <sub>2</sub>	[112, 23, 36] <sub>2</sub>	[96, 30, 24] <sub>2</sub>	[60, 26, 14] <sub>2</sub>
[130, 53, 24] <sub>2</sub>	[110, 45, 22] <sub>2</sub>	[92, 35, 20] <sub>2</sub>	[56, 19, 16] <sub>2</sub>
[128, 52, 24] <sub>2</sub>	[110, 41, 24] <sub>2</sub>	[92, 34, 20] <sub>2</sub>	[48, 21, 12] <sub>2</sub>
[128, 51, 24] <sub>2</sub>	[108, 49, 20] <sub>2</sub>	[90, 40, 18] <sub>2</sub>	

where  $G_2$  is a generator matrix of  $C_2$ ,  $G_3$  is a generator matrix of  $C_3$  and the rows of  $G_1^*$  are a set of linearly independent vectors of  $C_1$  that are not in  $C_2$  such that

$$\begin{bmatrix} G_1^* \\ G_2 \end{bmatrix}$$

generates  $C_1$ .

The main problem in applying ConstructionX is finding the best possible  $C_2$  while keeping  $b$  small. Since  $d_2$  is an upper bound, we want to maximize the minimum distance of  $C_2$ ; however, we also want a good minimum distance of  $C_3$  to maximize the lower bound, which ConstructionX does not often exceed. In fact, none of the ConstructionX codes presented in this paper exceed the lower bound, and each of them either have  $d = d_1 + d_3$  or  $d = d_2 = d_1 + d_3$ . None have  $d = d_2 \neq d_1 + d_3$  or  $d_2 \neq d \neq d_1 + d_3$ . Thus, want to pick a  $C_3$  of lower dimension, which will generally increase  $d_3$ . The challenge then becomes finding a good  $C_2$  of higher dimension, or in other words setting  $b$  to a small value. In our searches, we examined specifically  $1 \leq b \leq 6$ , and did not find any new BKLCs with  $b \geq 3$ .

In Daskalov and Hristov (2017), new record breaking codes over  $\mathbb{F}_3$  were found by looking at QC sub-codes of good QC codes. We implemented an algorithm to examine all QC sub-codes of a given dimension for a good QC code and choose the best one for use in ConstructionX, and additionally did the same for super-codes.

Our method is based on the following observations:

- Let  $C$  be a 1-generator QC code generated by  $(gf_1, gf_2, \dots, gf_\ell)$ . Then for each divisor  $p$  of  $g$ , the 1-generator QC code generated by  $(g'f_1, g'f_2, \dots, g'f_\ell)$  is a super-code of  $C$ , where  $g' = \frac{g}{p}$
- Let  $C$  be a 1-generator QC code generated by  $(gf_1, gf_2, \dots, gf_\ell)$  where  $x^m - 1 = gh$ . Then for each divisor  $p$  of  $h$ , the 1-generator QC code generated by  $(pgf_1, pgf_2, \dots, pgf_\ell)$  is a sub-code of  $C$ .

Using these facts, we are able to quickly examine QC sub- and super-codes for a given QC code. In our searches, we used QC codes with the parameters of BKLCs as our initial QC code, as well as the binary QC codes outlined above. Codes were pulled from the database at (Grassl 2022) and checked for being QC. If they were in fact QC codes, we used them in this algorithm. While  $g$  may be found through examining the inputs of the ASR algorithm, it is more generally found by taking the greatest common divisor of the set of generator polynomials. Our algorithm for super-code examination in psuedo-code is as follows:

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**Algorithm 1:** Finding ConstructionX Codes from Good QC Codes
 

---

```

Input:  $fs = [f_1, \dots, f_\ell]$ ;
Input:  $b$ ;
 $g = \text{gcd}(fs)$ ;
 $C = \text{QCcode}(fs)$ ;
while Factors of  $g$  remain do
  factor = The next factor of  $g$ ;
  if Degree(factor)  $\neq b$  then
    | continue;
  end
  newfs = [ $\frac{f}{\text{factor}}$  for  $f$  in  $fs$ ];
  superC = QCcode(newfs);
  if MinimumDistance(Best) < MinimumDistance(superC) then
    | Best = SuperC;
  end
end
for length to max do
  | C3 = BKLC(length,b);
  | CX = ConstructionX(C,Best,C3);
  | print(CX);
end
Result: New high minimum distance codes from ConstructionX
  
```

---

## 6 New BKLCs from ConstructionX

Using this method, we found 33 new record breaking linear codes over  $\mathbb{F}_2$ ,  $\mathbb{F}_3$ ,  $\mathbb{F}_4$  and  $\mathbb{F}_5$  according to the database at (Grassl 2022), including one new QC code which is a BKLC that was found incidentally.

**Theorem 6.1** *There exist linear codes with the following parameters:* [98, 30, 26]<sub>2</sub>, [97, 30, 25]<sub>2</sub>, [99, 31, 26]<sub>2</sub>, [98, 31, 25]<sub>2</sub>, [176, 51, 41]<sub>2</sub>, [177, 52, 41]<sub>2</sub>, [177, 51, 42]<sub>2</sub>, [178, 52, 42]<sub>2</sub>, [112, 23, 46]<sub>3</sub>, [100, 28, 35]<sub>3</sub>, [101, 26, 37]<sub>3</sub>, [105, 31, 35]<sub>3</sub>, [106, 23, 43]<sub>3</sub>, [107, 23, 43]<sub>3</sub>, [108, 31, 36]<sub>3</sub>, [113, 23, 47]<sub>3</sub>, [114, 23, 48]<sub>3</sub>, [115, 23, 48]<sub>3</sub>, [141, 26, 59]<sub>3</sub>, [164, 27, 70]<sub>3</sub>, [166, 27, 71]<sub>3</sub>, [167, 27, 72]<sub>3</sub>, [168, 27, 72]<sub>3</sub>, [169, 26, 74]<sub>3</sub>, [170, 26, 75]<sub>3</sub>, [170, 27, 73]<sub>3</sub>, [171, 26, 75]<sub>3</sub>, [172, 26, 75]<sub>3</sub>, [190, 24, 88]<sub>3</sub>, [217, 14, 121]<sub>3</sub>, [218, 14, 122]<sub>3</sub>, [219, 14, 123]<sub>3</sub>, [81, 18, 40]<sub>5</sub>

The parameters of the codes and the corresponding super-codes used in the construction of the new codes are:

**Table 4** Record breaking ConstructionX codes

New code	Original code	Super-code	Third code
[98, 30, 26] <sub>2</sub>	[96, 29, 26] <sub>2</sub>	[96, 30, 24] <sub>2</sub>	[2, 1, 2] <sub>2</sub>
[97, 30, 25] <sub>2</sub>	[96, 29, 26] <sub>2</sub>	[96, 30, 24] <sub>2</sub>	[1, 1, 1] <sub>2</sub>
[99, 31, 26] <sub>2</sub>	[96, 29, 26] <sub>2</sub>	[96, 31, 24] <sub>2</sub>	[3, 2, 2] <sub>2</sub>
[98, 31, 25] <sub>2</sub>	[96, 29, 26] <sub>2</sub>	[96, 31, 24] <sub>2</sub>	[2, 2, 1] <sub>2</sub>
[177, 52, 41] <sub>2</sub>	[170, 48, 42] <sub>2</sub>	[170, 52, 38] <sub>2</sub>	[7, 4, 3] <sub>2</sub>
[178, 52, 42] <sub>2</sub>	[170, 48, 42] <sub>2</sub>	[170, 52, 38] <sub>2</sub>	[8, 4, 4] <sub>2</sub>
[100, 28, 35] <sub>3</sub>	[99, 27, 35] <sub>3</sub>	[99, 28, 34] <sub>3</sub>	[1, 1, 1] <sub>3</sub>
[101, 26, 37] <sub>3</sub>	[99, 25, 37] <sub>3</sub>	[99, 26, 35] <sub>3</sub>	[2, 1, 2] <sub>3</sub>
[105, 31, 35] <sub>3</sub>	[104, 30, 35] <sub>3</sub>	[104, 31, 34] <sub>3</sub>	[1, 1, 1] <sub>3</sub>
[107, 23, 43] <sub>3</sub>	[104, 21, 43] <sub>3</sub>	[104, 23, 41] <sub>3</sub>	[3, 2, 2] <sub>3</sub>
[108, 31, 36] <sub>3</sub>	[104, 28, 37] <sub>3</sub>	[104, 31, 34] <sub>3</sub>	[4, 3, 2] <sub>3</sub>
[113, 23, 47] <sub>3</sub>	[112, 22, 48] <sub>3</sub>	[112, 23, 46] <sub>3</sub>	[1, 1, 1] <sub>3</sub>
[114, 23, 48] <sub>3</sub>	[112, 22, 48] <sub>3</sub>	[112, 23, 46] <sub>3</sub>	[2, 1, 2] <sub>3</sub>
[115, 23, 48] <sub>3</sub>	[112, 22, 48] <sub>3</sub>	[112, 23, 46] <sub>3</sub>	[3, 1, 3] <sub>3</sub>
[164, 27, 70] <sub>3</sub>	[160, 24, 72] <sub>3</sub>	[160, 27, 68] <sub>3</sub>	[4, 3, 2] <sub>3</sub>
[166, 27, 71] <sub>3</sub>	[160, 24, 72] <sub>3</sub>	[160, 27, 68] <sub>3</sub>	[6, 3, 3] <sub>3</sub>
[167, 27, 72] <sub>3</sub>	[160, 24, 72] <sub>3</sub>	[160, 27, 68] <sub>3</sub>	[7, 3, 4] <sub>3</sub>
[168, 27, 72] <sub>3</sub>	[160, 24, 72] <sub>3</sub>	[160, 27, 68] <sub>3</sub>	[8, 3, 5] <sub>3</sub>
[169, 26, 74] <sub>3</sub>	[160, 22, 75] <sub>3</sub>	[160, 26, 69] <sub>3</sub>	[9, 4, 5] <sub>3</sub>
[170, 26, 75] <sub>3</sub>	[160, 22, 75] <sub>3</sub>	[160, 26, 69] <sub>3</sub>	[10, 4, 6] <sub>3</sub>
[170, 27, 73] <sub>3</sub>	[160, 23, 73] <sub>3</sub>	[160, 27, 67] <sub>3</sub>	[10, 4, 6] <sub>3</sub>
[171, 26, 75] <sub>3</sub>	[160, 22, 75] <sub>3</sub>	[160, 26, 69] <sub>3</sub>	[11, 4, 6] <sub>3</sub>
[172, 26, 75] <sub>3</sub>	[160, 22, 75] <sub>3</sub>	[160, 26, 69] <sub>3</sub>	[12, 4, 6] <sub>3</sub>
[190, 24, 88] <sub>3</sub>	[182, 21, 88] <sub>3</sub>	[182, 24, 83] <sub>3</sub>	[8, 5, 3] <sub>3</sub>
[81, 18, 40] <sub>5</sub>	[78, 16, 40] <sub>5</sub>	[78, 18, 37] <sub>5</sub>	[3, 2, 2] <sub>5</sub>

**Table 5** Record breaking ConstructionX codes

New code	Original code	Sub-code	Third code
[141, 26, 59] <sub>3</sub>	[140, 26, 58] <sub>3</sub>	[140, 25, 59] <sub>3</sub>	[1, 1, 1] <sub>3</sub>
[217, 14, 121] <sub>3</sub>	[208, 14, 117] <sub>3</sub>	[208, 9, 126] <sub>3</sub>	[9, 5, 4] <sub>3</sub>
[218, 14, 122] <sub>3</sub>	[208, 14, 117] <sub>3</sub>	[208, 9, 126] <sub>3</sub>	[10, 5, 5] <sub>3</sub>
[219, 14, 123] <sub>3</sub>	[208, 14, 117] <sub>3</sub>	[208, 9, 126] <sub>3</sub>	[11, 5, 6] <sub>3</sub>

**Table 6** New record breakers by modification

New code	Original code	Modification method
[177, 51, 42] <sub>2</sub>	[177, 52, 41] <sub>2</sub>	Expurgation
[176, 51, 41] <sub>2</sub>	[177, 52, 41] <sub>2</sub>	Shorten at position 169
[106, 23, 43] <sub>3</sub>	[107, 23, 43] <sub>3</sub>	Puncture at position 106

**Table 7** QC codes which generate record breaking ConstX codes

Parameters	Generators
[96, 29, 26] <sub>2</sub>	[71, 64113173343, 27771046431]
[96, 30, 24] <sub>2</sub>	[5, 24076056631, 6252773246]
[96, 31, 24] <sub>2</sub>	[3, 6305574556, 2641521632]
[170, 48, 42] <sub>2</sub>	[100000000000000377323447615, 4135600657027030415272654623]
[170, 52, 38] <sub>2</sub>	[341603416034160314066672363, 427200026166224122662430431]
[140, 26, 58] <sub>3</sub>	[75656265543767850565651, 6273308468022430087634822263154874]
[140, 25, 59] <sub>3</sub>	[83434633507657031443433, 36887111402772710175802677383402711]
[99, 27, 35] <sub>3</sub>	[0300000000003141, 5151832674585001, 4201041431785337]
[99, 28, 34] <sub>3</sub>	[0688888888888505, 186426033663544, 5884464650276512]
[99, 25, 37] <sub>3</sub>	[00000100000385, 51172578486836861, 65173507073567432]
[99, 26, 35] <sub>3</sub>	[00000888888521, 1843867363731323, 3502668787557716]
[104, 30, 35] <sub>3</sub>	[0000000001000034635424162, 63878246257051483836350245]
[104, 31, 34] <sub>3</sub>	[000000000888885626614647, 3177381386786463175754811]
[112, 22, 48] <sub>3</sub>	[225217636715327031, 3855753628142726518311446808]
[112, 23, 46] <sub>3</sub>	[48608726228658785, 6355351383058703507508157342]
[104, 21, 43] <sub>3</sub>	[0000000036740414327721684, 13231000037464840407461361]
[104, 23, 41] <sub>3</sub>	[000000006051168412467604, 286355552403232277082171]
[104, 28, 37] <sub>3</sub>	[300000000000774187451744, 35747612345623613624005564]
[104, 31, 34] <sub>3</sub>	[654720654720653833336071, 634261474208860240816081]
[160, 24, 72] <sub>3</sub>	[100000000007321057462270530227344544615, 6588363577683773486427864331370312740533]
[160, 27, 68] <sub>3</sub>	[856085608560785553784776448685835748281, 350237067675375315443715352658617323401]
[160, 22, 75] <sub>3</sub>	[00030000004061832741081527113005224724, 763324150001553440888665474088827327074]
[160, 26, 69] <sub>3</sub>	[0003628805165221013063606507063164863, 784154000011055215802127766367524533]
[160, 23, 73] <sub>3</sub>	[0000003000068511234582485684775686524412, 632677162804764880141744613022227241818]
[160, 27, 67] <sub>3</sub>	[0000007032060245487053016568864183237, 54702682323740810856185340274102477412]
[208, 14, 117] <sub>3</sub>	[0748724866871624680580367184142241783262013113142526, 1245403380035706885414525207425072611683628176418272]
[208, 9, 126] <sub>3</sub>	[0431024638315531438181714874147403865036051473417760472, 707452166345144686117316063134645810071144055574570547]
[140, 18, 75] <sub>4</sub>	[101, <i>aababba1aa1b1b0baa</i> , 111110 <i>abab1a100b1a0b</i> , <i>b1b10b010b01ba1100ab</i> , 1110a0010aaa01 <i>bab10a</i> , <i>b1bab0baaaa0a1bbaabb</i> , <i>ba0baabbb0a0a1babb10</i> ]
[140, 19, 72] <sub>4</sub>	[11, <i>a0b1a1ba0ab010bb0a</i> , 101011 <i>0a10abbb01bb</i> , <i>ba100bbaa110b10111b</i> , 1011 <i>bbbaa0a001a0baa</i> , <i>ba1b00b1b1bb10b0a0b</i> , <i>b11a0a1a11bb10b1a10</i> ]
[78, 18, 40] <sub>5</sub>	[12312024143330311210411103220134021044, 111341132034241330331232130030204321433]
[78, 18, 37] <sub>5</sub>	[111424141431404200144344223204410104, 1003123311124341314340434104421210103]

We also found a few ternary record breakers by examining sub-codes as opposed to super-codes, and more record breakers were found by modifying the codes in Tables 4 and 5.

Additionally, upon submitting our record breakers for verification to Grassl (2022), the owner of the database, M. Grassl, informed us that he had found several new codes using our codes and a method known as ConstructionXX, which generalizes ConstructionX to utilize two additional codes and two sub-codes rather than just one. The parameters of these codes are given in Theorem 6.2.

**Theorem 6.2** *There exist linear codes with the following parameters:*  $[106, 23, 43]_3$ ,  $[109, 23, 44]_3$ ,  $[142, 26, 60]_3$ ,  $[167, 23, 77]_3$ ,  $[169, 23, 78]_3$ ,  $[171, 27, 74]_3$ ,  $[175, 26, 77]_3$ ,  $[177, 26, 78]_3$ .

Finally, the table below presents the generators for the original and super/sub-codes that were used as components of the new ConstructionX code. Each of the codes in the table corresponds to a set of parameters found in Table 4 or Table 5.

The binary codes are presented in the same way as the previous tables, and the ternary codes are presented with the same algorithm, but instead of converting 3 places to base 8, it converts 2 places to base 9. The  $\mathbb{F}_4$ -codes are presented unabridged and with  $b = a^2$ . The codes over  $\mathbb{F}_5$  are also presented unabridged. See Tables 6 and 7.

Since there are a few codes in this table which share parameters while being non-equivalent, each code is presented adjacent to its respective sub/super-code, and in approximately the same order as the table in which they appear. One of the QC super-codes, with parameters  $[112, 23, 46]_3$ , in the table below is also a record breaker in its own right.

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